AUB-CMPS 211: *Discrete Structures* Solutions February 17th, 2016

Quiz 1

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Duration: 1 hour

This exam is closed notes.

Question 1 (25%)

Using symbolic derivation, show that $(p \to r) \wedge (q \to r)$ is logically equivalent to $(p \vee q) \to r$. Justify each of your steps.

Solution:

$[(p \vee q) \to r] \to [(p \to r) \wedge (q \to r)]$:

1.	$\equiv (p \vee q) \to r$	Premise	
2.	$\equiv \neg(p \vee q) \vee r$	Resolution from 1	(6.25 pts)
3.	$\equiv (\neg p \wedge \neg q) \vee r$	De Morgan from 2	(6.25 pts)
4.	$\equiv (\neg p \vee r) \wedge (\neg q \vee r))$	Distribution from 3	(6.25 pts)
5.	$\equiv (p \to r) \wedge (q \to r))$	Resolution from 4	(6.25 pts)

Question 2 (25%)

Translate each of the below sentences into predicate logic.

1. There is no one who can fool everybody.
2. No one can fool both Fred and Jerry.
3. Nancy can fool exactly two people.

Repeat by translating the negation of each of the sentences above into predicate logic.

Solution:

Define: fool(x, y): x can fool y, where the domain of x and y is the set of all people.

1. *Statement:* There is no one who can fool everybody.
   Predicate: $\forall x \exists y \ \neg \text{fool}(x, y)$ (3 pts)
   English Negation: Someone can fool everyone. (2 pts)
   Negation: $\exists x \forall y \ \text{fool}(x, y)$ (2 pts)

2. *Statement:* No one can fool both Fred and Jerry.
   Predicate: $\forall x(\neg \text{fool}(x, Fred) \vee \neg \text{fool}(x, Jerry))$ (3 pts)
   English Negation: Someone can fool both Fred and Jerry. (2 pts)
   Negation: $\exists x(\text{fool}(x, Fred) \wedge \text{fool}(x, Jerry)$ (3 pts)

3. *Statement:* Nancy can fool exactly two people.
   Predicate: $\exists x_1, x_2(x_1 \neq x_2 \wedge \text{fool}(Nancy, x_1) \wedge \text{fool}(Nancy, x_2) \wedge \forall y(y \neq x_1 \wedge y \neq x_2 \rightarrow \neg \text{fool}(Nancy, y)))$ (4 pts)
   English Negation: (2 pts)

   (a) It is not the case that Nancy can fool exactly two people
   (b) For any 2 distinct people, those are not the two and only two people Nancy can fool
   (c) For any pair of people, if Nancy can fool both of them, then she can fool another pair

   Negation: (4 pts)

   (a) $\forall x_1, x_2\,(x_1 \neq x_2 \rightarrow (\neg fool(Nancy, x_1) \vee \neg fool(Nancy, x_2) \vee \exists y(fool(Nancy, y) \wedge y \neq x_1 \wedge y \neq x_2)))$

Question 3 (25%)

Show that the premises

- All insects have six legs
- Dragon Flies are insects
- Spiders do not have six legs

lead to the conclusion

- Spiders are not insects.

Justify each of your steps, and consider the domain of discourse to be the set of all animals.

Solution:

The domain throughout the discourse is the set of all animals. The statement "Dragon Flies are insects" is not used. Let $I(x)$ = "x is an insect", $L(x)$ = "x has six legs", $S(x)$ = "x is a spider". The two premises of relevance are

$$\forall x\,(I(x) \rightarrow L(x)) \qquad \text{(3 pts)}$$

and

$$\forall x\,(S(x) \rightarrow \neg L(x)) \qquad \text{(3 pts)}$$

We now have:

1.	$\forall x\,(I(x) \to L(x))$	Premise (2 pts)
2.	$I(c) \to L(c)$	U.I. from 1 (3 pts)
3.	$\forall x\,(S(x) \to \neg L(x))$	Premise (2 pts)
4.	$S(c) \to \neg L(c)$	U.I. from 3 (3 pts)
5.	$\neg L(c) \to \neg I(c)$	Contraposition of 4 (3 pts)
6.	$S(c) \to \neg I(c)$	Hypothetical Syllogism from 5 (3 pts)
7.	$\forall x\,(S(x) \to \neg I(x))$	U. G. from 6 (3 pts)

Question 4 (25%)

Prove or disprove. Justify using tools from formal logic (no credit will be given otherwise):

- If m and n are real numbers whose average is irrational, then m is irrational or n is irrational.

Solution:

This is a universtal claim (1 pt)

Proof by Contra-position (or contradiction) (2 pts)

Assume m and n are both rationals, and write (1.5 pts)

$$m = \frac{a}{b}; a, b \in Z \wedge b \neq 0; gcd(a, b) = 1$$

$$n = \frac{c}{d}; c, d \in Z \wedge d \neq 0, gcd(c, d) = 1$$

The average of m and n is equal to (2 pts)

$$\frac{m+n}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{\frac{ad+cb}{db}}{2} = \frac{ad+cb}{2db}$$

Since a, b, c, d are all integers and Z is closed under multiplication and addition, then $ad, cb, ad + cb, db, 2db$ are also integers. (2 pts)
Since $b \neq 0$ and $d \neq 0$ then $2db \neq 0$ (2 pts)

Therefore $\frac{ad+cb}{2db} = \frac{m+n}{2}$ is rational. (2 pts)

$\square$

- There are no solutions in positive integers x and y to the equation $x^2 + y^2 = 15$.

Solution:

This claim is universal. (1 pt)
If we have $x \geq 4$ then $x^2 \geq 16$ and $y^2 \geq 0$, for which $x^2 + y^2 \geq 16$ (2 pt)
Else, if we have $y \geq 4$ then $y^2 \geq 16$, and $x^2 \geq 0$, for which $y^2 + x^2 \geq 16$ (2 pt)
We will thus require $0 < x, y < 4$ *(x, y are both positive integers from the given)* and so the u.o.d. is $\{1, 2, 3\}$ (2 pt)
Since the u.o.d. is countable and finite, we will use an exhaustive proof (1.5 pt)

Without loss of generality we can assume that $x \geq y$ since addition commutative.

(3 pts for all six checks below):

1. $x = 1, y = 1$ then $x^2 + y^2 = 2 \neq 15$
2. $x = 2, y = 1$ then $x^2 + y^2 = 5 \neq 15$
3. $x = 2, y = 2$ then $x^2 + y^2 = 8 \neq 15$
4. $x = 3, y = 1$ then $x^2 + y^2 = 10 \neq 15$
5. $x = 3, y = 2$ then $x^2 + y^2 = 13 \neq 15$
6. $x = 3, y = 3$ then $x^2 + y^2 = 18 \neq 15$

Therefore, There are no solutions for $x^2 + y^2 = 15$ in the positive integers [1] (1 pt).

□

[1] An interesting read on the topic of the sum of two squares:
https://en.wikipedia.org/wiki/Fermat's_theorem_on_sums_of_two_squares